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Cardy-Verlinde Formula and Thermodynamics of Black Hole in Higher Dimensional Space-Time

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In this paper we discuss thermodynamics parameters of black hole horizon and cosmological horizon in general high-dimensional space-time. We obtain that the entropy of a cosmological horizon can be described by the Cardy-Verlinde formula. However, the entropy of black hole horizon will be expressed in a form of the Cardy-Verlinde formula, if one adopts the methods given by Abbott and Deser to compute the mass of a black hole in general high-dimensional space-time. Through discussion, relation among various thermodynamics parameters of the black hole in general high-dimensional space-time is given. That is, differential formula of the first law of thermodynamics is obtained. Because we discuss the general high-dimensional space-time, our result has universality.

KEY WORDS: Cardy-Verlinde formula; casimir energy; high-dimensional space-time.

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1. INTRODUCTION

Entropy of the black hole is one of the important subjects in theoretical physics. Since entropy has statistical meaning, the understanding of entropy involves the sense of the microscopic essence of the black hole. Fully understanding of it needs a good quantum gravitation theory. It is thought that an efficient theory of quantum gravitation should contain the definition of Bekenstein-Hawking entropy in its frame. However, at present the work of it is not satisfying. The statistical origin of the black hole is not solved yet (Liberati, 1997). dS/CFT seize on key in the microcosmic interpretation of the black hole entropy. Cardy-Verlinde formula given by Verlinde is the relation among entropy, total energy of some conformal field in arbitrary dimension space-time and Casimir energy. This formula is valid for different black holes.

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Recently much attention has been focused on studying de Sitter (dS) space and asymptotically dS space. This is motivated at least by the following aspect. Recent analysis of astronomical data for supernova indicates that there is a positive cosmological constant in our universe (Perlmutter *et al.*, 1997; Caldwell *et al.*, 1998; Carnavich *et al.*, 1998). Thus our universe might approach to a dS phase in the far future. One of the important subjects in theory physics is studying thermodynamics properties of dS space-time. Li and Shen (2004), Li and Zhao (2001), Gao and Shen (2003) and Zhang and Zhao (2004) computed the entropy of dS space-time in terms of brick-wall method. Zhao *et al.* (2002, 2003, 2004) investigated the statistical entropy of dS space-time via the membrane model. They all derived more significant results. Recently, Cai (2002a,b), Myung (2002), and Setare and Altaie (2003) discussed thermodynamics parameters of dS space-time and calculated Casimir energy. The entropy of black hole horizon and the entropy of cosmological horizons can also be expressed in terms of the Cardy-Verlinde formula.

In this paper, we will generalize the discussion in Cai (2002a) to the case of general high-dimensional space-time. For general high-dimensional space-time, except for the cosmological horizon, there is a black hole horizon, which has also associated Hawking radiation and entropy with different temperature, if we adopt the definition of mass due to Balasubranmanian, de Boer and Minic (BBM), (Balasubranmanian *et al.*, 2002), we obtain that the entropy of black hole horizon can not be expressed in a form of the Cardy-Verlinde formula. But if we use the method of Abbott and Deser (1982) in general high-dimensional space-time, we can derive that the entropy of black hole horizon can be expressed in a form of the Cardy-Verlinde formula. Therefore, we obtain the thermodynamics first law expression of general high-dimensional black hole. In Section 4 we will give its application.

2. ENTROPY OF THE COSMOLOGICAL HORIZON CARDY-VERLINDE FORMULA

The linear element of high-dimensional space-time is given by Gallo (2004)

$$dS^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2 d\Omega_{d-2}^2, \tag{1}$$

here

$$N(r) = A^{-1}(r) = \left\{1 - \frac{2m(r)}{r^{d-3}}\right\}^{1/2}, \tag{2}$$

where

$$m(r) = M + \frac{\Lambda r^{d-1}}{(d-2)(d-1)} - \frac{8\pi C r^{[(d-2)\lambda+1]}}{(d-2)[(d-2)\lambda+1]}$$
$$\text{if}\quad \lambda \neq -\frac{1}{d-2};\quad C \neq 0$$

Λ is cosmological constant. Space-time (1) has two horizons. There are cosmological horizon location r_{c} and black hole horizon location r_+. And it satisfies equation $N(r) = 0$.

The cosmological horizon has associated thermodynamic quantities

$$
\begin{aligned}
T &= \frac{1}{4\pi r_{\mathrm{c}}}\left\{-(d-3) - 2\frac{8\pi C}{d-2} r_{\mathrm{c}}^{[(d-2)\lambda-d+4]} + \frac{2\Lambda r_{\mathrm{c}}^2}{(d-2)}\right\},\\
S &= \frac{r_{\mathrm{c}}^{d-2}\mathrm{Vol}\,(S^{d-2})}{4G},\\
\phi_m &= \frac{\partial E_{\mathrm{tot}}}{\partial C} = -\frac{8\pi r_{\mathrm{c}}^{[(d-2)\lambda+1]}}{(d-2)[(d-2)\lambda+1]},
\end{aligned} \tag{3}
$$

where G is d-dimensional gravitation constant, $\mathrm{Vol}(S^{d-2})$ denotes the volume of unit $(d-2)$ sphere $d\Omega^2_{d-2}$, ϕ_m is potential conjugate to C. Following Youm (2001), Cai (2001) and Klemm *et al.* (2002), we define the zero temperature contribution, called the proper internal energy, as

$$
E_q = 8\pi C r_{\mathrm{c}}^{[d-2)\lambda+1]} \frac{1+\lambda}{(d-2)\lambda+1}. \tag{4}
$$

So, the thermal excitation energy is

$$
E = E_{\mathrm{tot}} - E_q, \tag{5}
$$

where, E_{tot} is Balasubranmanian, de Beerand and Minic (BBM), (Balasubranmanian *et al.*, 2002) mass of black hole

$$
E_{\mathrm{tot}} = -M = \frac{\Lambda r_{\mathrm{c}}^{d-1}}{(d-2)(d-1)} - \frac{8\pi C r_{\mathrm{c}}^{[(d-2\lambda+1]}}{(d-2)[(d-2)\lambda+1]} - \frac{r_{\mathrm{c}}^{d-3}}{2}. \tag{6}
$$

If we assume that the holographic dual theory is conformal, then the pressure $p = -(\frac{\partial E}{\partial V})_{S,C}$ is given by

$$
p = \frac{E}{(d-2)V}, \tag{7}
$$

where V is the volume of the system. So, the Casimir energy of the holographic dual theory is given by

$$
E_{\mathrm{c}} = (d-2)(E_{\mathrm{tot}} + pV - TS - 2\phi_m C) = -\frac{\mathrm{Vol}(S^{d-2})}{8\pi G}(d-2) r_{\mathrm{c}}^{d-3}. \tag{8}
$$

So the entropy corresponding cosmological horizons (3) can be written as

$$
S = \frac{2\pi l}{d-2}\sqrt{|E_{\mathrm{c}}|\,(2(E_{\mathrm{tot}} - E_m) - E_{\mathrm{c}})}, \tag{9}
$$

where

$$E_m = \phi_m C = -\frac{8C\pi r_{\rm c}^{[(d-2)\lambda+1]}}{(d-2)[(d-2)\lambda+1]} \tag{10}$$

$$l^2 = \frac{(d-1)(d-2)}{2\Lambda}, \quad \frac{Vol(S^{d-2})}{G} = \frac{8\pi}{d-2}. \tag{11}$$

This result shows that the thermodynamics parameters of the cosmological horizon in static higher dimensional space-times can be described by conformal field, which is supposed to be an entropy formula of CFT in any dimension. Thus our result provides support of the dS/CFT correspondence. One can read Hull (2000), Balasubramanian *et al.* (2001), Witten (1998), and Mazur and Mottola (2001) for literature review.

From (3), we derive that thermodynamics functions corresponding the cosmological horizon satisfy the first law of thermodynamics

$$dE_{\rm tot} = TdS + \phi_m dC, \tag{12}$$

and

$$T = \left(\frac{\partial E_{\rm tot}}{\partial S}\right)_C, \quad \phi_m = \left(\frac{\partial E_{\rm tot}}{\partial C}\right)_S. \tag{13}$$

3. ENTROPY OF THE BLACK HOLE HORIZON AND CARDY-VERLINDE FORMULA

Thermodynamics parameters corresponding the black hole horizon are

$$\begin{aligned}\tilde{T} &= \frac{1}{4\pi r_+}\left\{(d-3) + 2\frac{8\pi C}{d-2} r_+^{[(d-2)\lambda-d+4]} - \frac{2\Lambda r_+^2}{(d-2)}\right\},\\ \tilde{S} &= \frac{r_+^{d-2} Vol(S^{d-2})}{4G},\\ \tilde{\phi}_m &= \frac{\partial \tilde{E}_{\rm tot}}{\partial C} = \frac{8\pi r_{\rm c}^{[(d-2)\lambda+1]}}{(d-2)[(d-2)\lambda+1]},\end{aligned} \tag{14}$$

Abbott and Deser (1982) mass

$$\tilde{E}_{\rm tot} = M = -\frac{\Lambda r_+^{d-1}}{(d-2)(d-1)} + \frac{8\pi C r_+^{[(d-2\lambda+1]}}{(d-2)[(d-2)\lambda+1]} - \frac{r_+^{d-3}}{2}. \tag{15}$$

We define the zero temperature contribution, called the proper internal energy, as

$$\tilde{E}_q = -8\pi C r_+^{[2-2)\lambda+1]} \frac{1+\lambda}{(d-2)\lambda+1}. \tag{16}$$

$$\tilde{E} = \tilde{E}_{\rm tot} - \tilde{E}_q, \quad p = \frac{\tilde{E}}{(d-2)V}, \tag{17}$$

Casimir energy is defined as

$$\tilde{E}_{\rm c} = (d-2)(\tilde{E}_{\rm tot} + pV - \tilde{T}\tilde{S} - 2\phi_m C) = \frac{{\rm Vol}(S^{d-2})}{8\pi G}(d-2)r_+^{d-3}.$$

So the entropy of the black hole horizon (14) can be written as

$$\tilde{S} = \frac{2\pi l}{d-2}\sqrt{\tilde{E}_{\rm c}(2(\tilde{E}_{\rm tot} - \tilde{E}_m) - \tilde{E}_{\rm c})} \tag{18}$$

where

$$\tilde{E}_m = \tilde{\phi}_m C = \frac{8C\pi r_{\rm c}^{[(d-2)\lambda+1]}}{(d-2)[(d-2)\lambda+1]}. \tag{19}$$

From (14), we derive that thermodynamics functions corresponding the black hole horizon satisfy the first law of thermodynamics

$$d\tilde{E}_{\rm tot} = \tilde{T}d\tilde{S} + \tilde{\phi}_m dC, \tag{20}$$

and

$$\tilde{T} = \left(\frac{\partial \tilde{E}}{\partial \tilde{S}}\right)_C, \quad \tilde{\phi}_m = \left(\frac{\partial \tilde{E}}{\partial C}\right)_{\tilde{S}}. \tag{21}$$

According to $\tilde{E}_{\rm tot} = -E_{\rm tot} = M$, from (12) and (20), we have

$$\tilde{T}d\tilde{S} + TdS + \tilde{\phi}_m dC + \phi_m dC = 0. \tag{22}$$

4. GIVE AN EXAMPLE

For Reissner-Nordstrom(dS/AdS) black holes.

$$\lambda = -1, \quad C = -\frac{Q^2}{8\pi}. \tag{23}$$

Thus

$$m(r) = M + \frac{\Lambda r^{d-1}}{(d-2)(d-1)} + \frac{Q^2}{(d-2)(d-3)r^{d-3}}. \tag{24}$$

And that

$$E_q = 0 \cdot E_m = -\frac{Q^2}{(d-2)(d-3)r_{\rm c}^{d-3}}, \quad E_{\rm c} = -\frac{{\rm Vol}(S^{d-2})}{8\pi G}(d-2)r_{\rm c}^{d-3},$$

$$E_{\rm tot} = -M = \frac{\Lambda r_{\rm c}^{d-1}}{(d-2)(d-1)} - \frac{Q^2}{(d-2)(d-3)r_{\rm c}^{d-3}} - \frac{r_{\rm c}^{d-3}}{2}. \tag{25}$$

Substituting (25) into (9), we derive for Reissner-Nordstrom(dS/AdS) black holes the entropy of the cosmological horizon can be expressed as

$$S = \frac{2\pi l}{d-2}\sqrt{|E_{\mathrm{c}}|\,(2(E_{\mathrm{tot}} - E_m) - E_{\mathrm{c}})}. \tag{26}$$

In similar way, the entropy of the black hole horizon can be expressed as

$$\tilde{S} = \frac{2\pi l}{d-2}\sqrt{\tilde{E}_{\mathrm{c}}(2(\tilde{E}_{\mathrm{tot}} - \tilde{E}_m) - \tilde{E}_{\mathrm{c}})}. \tag{27}$$

For d-dimensional monopole black hole

$$\lambda = \frac{d-4}{d-2}, \quad C = -\frac{(d-2)\eta^2}{2}, \tag{28}$$

thus

$$m(r) = M + \frac{\Lambda r^{d-1}}{(d-2)(d-1)} + \frac{4\pi\eta^2}{(d-3)r^{d-3}}. \tag{29}$$

$$\begin{aligned}
E_q &= -8\pi\eta^2 r_{\mathrm{c}}^{d-3}, \quad E_{\mathrm{c}} = -\frac{\mathrm{Vol}(S^{d-2})}{8\pi G}(d-2)r_{\mathrm{c}}^{d-3},\\
E_m &= \frac{1}{2}\phi_m\eta = -\frac{4\pi\eta^2 r_{\mathrm{c}}^{d-3}}{d-3},\\
E_{\mathrm{tot}} &= -M = \frac{\Lambda r_{\mathrm{c}}^{d-1}}{(d-2)(d-1)} + \frac{4\pi\eta^2 r_{\mathrm{c}}^{d-3}}{(d-3)} - \frac{r_{\mathrm{c}}^{d-3}}{2}.
\end{aligned} \tag{30}$$

So the entropy of the cosmological horizon can be written as

$$S = \frac{2\pi l}{d-2}\sqrt{|E_{\mathrm{c}}|\,(2(E_{\mathrm{tot}} - E_m) - E_{\mathrm{c}})}, \tag{31}$$

where

$$E_m = \frac{1}{2}\phi_m\eta = -\frac{4\pi\eta^2 r_{\mathrm{c}}^{d-3}}{d-3}, \tag{32}$$

In similar way, the entropy of the black hole horizon can be expressed as

$$\tilde{S} = \frac{2\pi l}{d-2}\sqrt{\tilde{E}_{\mathrm{c}}(2(\tilde{E}_{\mathrm{tot}} - \tilde{E}_m) - \tilde{E}_{\mathrm{c}})}, \tag{33}$$

where

$$\tilde{E}_m = \frac{1}{2}\tilde{\phi}_m\eta = \frac{4\pi\eta^2 r_+^{d-3}}{d-3}, \quad \tilde{E}_{\mathrm{c}} = \frac{\mathrm{Vol}(S^{d-2})}{8\pi G}(d-2)r_+^{d-3}, \quad \tilde{E}_q = 8\pi\eta^2 r_+^{d-3}. \tag{34}$$

5. CONCLUSION

Based on the above discussion, for general high-dimensional space-times, we obtain that total entropy is the sum of the entropy corresponding to black hole horizon and the entropy corresponding to cosmological horizon. The entropy of a cosmological horizon can be described by the Cardy-Verlinde formula. If one uses the BBM mass of asymptotically dS space, the black hole horizon entropy can not be expressed by a form like the Cardy-Verlinde formula. Here we report that if we adopt the theory given in Abbott and Deser (1982) and calculate mass of general high-dimensional black holes by AD method , the black hole entropy can also be rewritten in a Cardy-Verlinde form. It is obtained that thermodynamics parameter of the black hole horizon can be described by CFT in general high-dimensional space-time. This result provides support for dS/CFT. Because we discuss the general high-dimensional space-time, our result has universality.

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REFERENCES

Abbott, L. F. and Deser, S. (1982). *Nucl. Phys. B* **195**, 76.
Balasubramanian, V., de Boe, J., and Minic, D. (2002). *Phys. Rev. D* **65**, 123508.
Balasubramanian, V., Horava, P., and Mininc, D. (2001). *JHEP* **0105**, 043.
Cai, R. G. (2001). *Phys. Rev. D* **63**, 124018.
Cai, R. G. (2002a). *Nucl. Phys. B* **628**, 375.
Cai, R. G. (2002b). *Phys. Lett. B* **525**, 331.
Caldwell, R. R., Dave, R., and Steinhardt, P. J. (1998). *Phys. Rev. Lett.* **80**, 1582.
Carnavich, P. M., Jha, S., Challis, P., *et al.* (1998). *Astrophys. J.* **509**, 74.
Gallo, E. (2004). *Gen. Rel. Grav.* **36**, 1463.
Gao, C. J. and Shen, Y. G. (2003). *Chin. Phys. Lett.* **20**, 618.
Hull, C. M. (2000). *Nucl. Phys. B* **575**, 231.
Klemm, D., Petkou, A. C., Siopsis, G., and Zanon, D. (2002). *Nucl. Phys. B* **620**, 519.
Liberati, S. (1997). *IL Nuovo Cimento. B* **112**, 405.
Li, X. and Shen, Y. G. (2004). *Gen. Rel. Grav.* **36**, 1473.
Li, X. and Zhao, Z. (2001). *Chin. Phys. Lett.* **18**, 463.
Mazur, P. O. and Mottola, E. (2001). *Phys. Rev. D* **64**, 104022.
Myung, Y. S. (2002). *Phys. Lett. B* **531**, 1.
Perlmutter, S., Gabi, S., Goldhaber, G., *et al.* (1997). *Astrophys. J.* **483**, 565.
Setare, M. R. and Altaie, M. B. (2003). *Eur. Phys. J. C* **30**, 273.
Witten, E. (1998). *Theor. Math. Phys.* **2**, 505.
Youm, D. (2001). *Mod. Phys. Lett. A* **16**, 1263.

Zhao, R., Wu, Y. Q., and Zhang, L. C. (2003). *Class. Quant. Grav.* **20**, 4885.
Zhao, R., Wu, Y. Q., and Zhang, L. C. (2004). *Chin. Phys.* **13**, 974.
Zhang, L. C. and Zhao, R. (2004). *Acta Phys. Sin.* **53**, 362 (in Chinese).
Zhao, R., Zhang, J. F., and Zhang, L. C. (2002). *Commun. Theor. Phys. (Beijing, China)* **37**, 4575.